Club Rates may be had on Application

Mathematics News Letter

SUBSCRIPTION $1.00 PER YEAR Single Copies . . 15c

Published eight times per year at Baton Rouge, Louisiana.

Dedicated to mathematics in general and to the following aims in particular: (1) a study of the common problems of secondary and collegiate mathematics teaching, (2) a true valuation of the disciplines of mathematics, (3) the publication of high class expository papers on mathematics, (4) the development of greater public interest in mathematics by the publication of authoritative papers treating its cultural, humanistic and historical phases.

VOL. 8 BATON ROUGE, LA., MARCH, 1934 No. 6

CONTENTS

The Philosopher and Mathematics

Early in the nineteenth century Thomas DeQuincey, brilliant English critic, essayist and philosopher, wrote as follows in his "Letters to a Young Man":[1] "Now you allege (I suppose upon occasion of my references to mathematics in my last letter) that you have no 'genius' for mathematics; and you speak with the usual awe of the supposed 'profundity' of intellect necessary to a great progress in this direction. Be assured that you are in utter error; though it be an error all but universal. In mathematics, upon two irresistible arguments which I shall set in a clear light, when I come to explain the procedure of the mind with regard to that sort of evidence, there can be no subtlety; all minds are levelled except as to the rapidity of the course; and from the entire absence of all those acts of mind which do really imply profundity of intellect, it is a question whether an idiot might not be made an excellent mathematician. Listen not to the romantic notions of the world on this subject; above all listen not to the mathematicians. Mathematicians as *mathematicians*, have no business with the question. With respect to this it is memorable, that in no one of the great philosophical questions which the ascent of mathematics has from time to time brought, up above the horizon of our speculative view, has any mathematician who was merely such had depth of intellect adequate to its solution . . . I need only ask what English or French mathematician has been able to exhibit the notion of *negative quantities* in a theory endurable even to a popular philosophy or which has commanded any assent? . . . But, not to trouble you with more of these cases so opprobrious to mathematicians, lay this to heart, that mathematics is very easy and very important; in fact it is the organ of one large division of human knowledge."

One finds in the same "Letter" a second DeQuincey slant on mathematics. Though tinged with a suspicion of being somewhat contradictory to the first mentioned one, this second one is more comforting to the mere mathematician, who is usually indisposed to view mathematics as easy, though willing to concede its importance. DeQuincey's admiration of Leibnitz was hearty. Rightly did he esteem him as both philosopher and mathematician. He says: "Leibnitz, however anxious to throw out his mind upon the whole encyclopedia of human research, yet did not forget to pay the price at which only any *right* to be thus discursive can be earned: he sacrificed to the austerer muses; knowing that God geometrizes eternally, he rightly supposed that, in the universal temple, Mathesis (Mathematics) must furnish the master-key which would open most shrines. The Englishman,[2] on the contrary, I remarked to have been too self-indulgent and almost a voluptuary in his studies; sparing himself all toil and

[1] Ticknor, Reed and Fields, Boston, 1854.
[2] The context does not disclose his name.

thinking apparently to evade the necessity of artificial power by an extraordinary exertion of his own native power. Neither as a boy nor as a man had he submitted to any regular study or discipline of thought. His choice of subjects had not been amongst those which admitted of *continuous* thinking and study."

One, who for the first time notes this critical aggressiveness of DeQuincey toward mathematics, its nature and limitations, must reasonably raise the question, by what measure of mathematical experience or knowledge did he become qualified as a mathematical critic? Scattered throughout his published essays are numerous notes startling in their union of perfectly blended philosophy and mathematics. Here is a sample: [3]"And the commonest reader will understand what I mean, when I tell him, that if it were possible for the relation between the square and the circle (i. e., between the diameter and the circumference) to be assigned exactly, and not (as it now is) infinitely near,—the consequences would be, not merely (as he supposes) that a mind had arisen which saw what had escaped all former minds, but also, that, for the first time an internal war would arise in mathematics: antinomies would be established: A and non-A would be equally true; contradictory positions would co-exist; in short the supposed discovery would be inconsistent with existing truths."

It was at the Koningsberg University that Immanuel Kant, greatest of German philosophers, one of the greatest in all time, studied philosophy and mathematics from 1740 to 1746. In his "Critique of Human Reason",[4] under the fundamental heading of transcendental logic, are found in his proofs of "these" and "antitheses" all the forms, trappings and spirit of the mathematical process. Kant says: "The brilliant claims of reason striving to extend its dominion beyond the limits of experience have been represented above only in dry formulae, which contain merely the grounds of its pretensions . . . The questions: whether the world has a beginning and a limit to its extension in space; whether there exists anywhere, or perhaps, in my own thinking Self an indivisible and indestructible unity—or whether nothing but what is divisible and transitory exists; whether I am a free agent, or, like other beings, am bound in the chains of nature and fate; whether, finally, there is a supreme cause of the world, or all our thought and speculation must end with nature and the order of external things—are questions, for the solution of which the mathematician would willingly exchange his whole science; for in it there is no satisfaction for the highest aspirations and most ardent desires of humanity. Nay, it may even be said that the true value of mathematics—that pride of human reason —consists in this: that she guides reason to the knowledge of nature— in her greater, as well as in her less manifestations—in her beautiful

[3] Essay on Kant.

[4] Translated into English by J. M. D. Meiklejohn. Published by P. F. Collier and Son, New York, 1901.

order and regularity—guides her, moreover, to an insight into the wonderful unity of the moving forces in the operations of nature, far beyond the expectations of a philosophy building only on experience; and that she (mathematics) thus encourages philosophy to extend the province of reason beyond all experience, and at the same time provides it with the most excellent materials for supporting its investigations, in so far as their nature admits, by adequate and accordant intuitions."

Herbert Spencer, founder of the "System of Synthetic Philosophy," the most gigantic intellectual product of the 19th century, like many others of polyphase genius was mathematician before he was philosopher. Apparently with no interest in pursuing an academic degree, he yet devoted the leisure time of his 'teens and early twenties to mathematics and the work of a civil engineer. One easily conjectures from a study of his development that in early manhood his faculties craved the unlimited athletics to be found in mathematics, that as his intellect further evolved it passed to engineering or other applications of mathematical science, from which in a final evolution it emerged to embrace the problem of problems—one whose solution would furnish a generalization of the "complete history of the knowable universe."

Rene Descartes (1596-1650), French philosopher, gave himself up to mathematical investigations from 1614 to 1616. His nurture of the scheme of a wedded algebra and geometry was the first of those urges whose continued pressure was destined finally to break the mathematical chrysalis out of which was to emerge the philosopher. By 1625, at the age of 29, he "had ceased to devote himself to pure mathematics" and was beginning to turn his studies to the "nature of man, of the soul and of God." It is possible that a De Quincey slant on mathematics might have pronounced that not the mathematician alone but the mathematician and philosopher, united in Descartes, created that beautiful branch of mathematics now known as analytical geometry.

Twenty-two years before Herbert Spencer, came Auguste Comte, founder of the second greatest philosophical system of the 19th century. In his Positive Philosophy he sets up a hierarchy of the sciences, whose order now classic is as follows: (1) Mathematics, (2) Astronomy, (3) Physics, (4) Chemistry, (5) Biology, (6) Sociology. As true to form as were Descartes' and Spencer's, his own mind grew through mathematics to mathematics plus philosophy. At eighteen he was teaching the latter for a living. At twenty-eight he was lecturing on his Positive Philosophy.[5] Significant it was that these lectures were listened to by physicist, biologist and mathematician.

That portion of his Positive Philosophy which Comte called the [6]Philosophy of Mathematics was pronounced by James Mill to be

[5]Article on Comte in Encyclopedia Britannica.

[6]Translated from the French by W. M. Gillespie. Published separately by Harper and Brothers, New York, 1851.

"by far the greatest yet produced on the philosophy of the sciences." Though uttered by Comte nearly a century ago the following words pay a far more glorious tribute to mathematics than Kant had paid it: "It is indispensable to perceive, first of all, that, in the purely logical point of view, this science is by itself necessarily and rigorously universal; for there is no question whatever which may not be finally conceived as consisting in determining certain quantities from others by means of certain relations, and consequently as admitting of reduction, in final analysis, to a simple question of numbers. In all our researches, indeed, on whatever subject, our object is to arrive at numbers, at quantities, though often in a very imperfect manner and by very uncertain methods. Thus, taking an example in the class of subjects the least accessible to mathematics, the phenomena of living bodies, even when considered (to take the most complicated case) in the state of disease, is it not manifest that all the questions of therapeutics may be viewed as consisting in determining the quantities of the different agents which modify the organism, and which must act upon it to bring it to its normal state, admitting, for some of these quantities in certain cases, values which are equal to zero, or negative, or even contradictory?"

Of Bertrand Russell foremost present living example of combined mathematician and philosopher we may write later.

—S. T. S.

A Science of Education Proposed

By W. Paul Webber

It is an observed fact that some teachers of college mathematics make the subject so laborious and unfruitful of contacts with other subjects that sometimes students who have an inclination for mathematics are repelled or discouraged to the extent that they avoid courses in mathematics after such an experience, and especially do they avoid that type of teacher in their future work. We assert that it is reasonable that mathematics (or any subject) should be laid before a class in as attractive form as the circumstances permit. This does not mean that artificial devices for avoiding thoroughness shall be employed. But it does mean that the view point of class work should encourage and help students to see how mathematics is clearly useful and enjoyable. Until students get such a view point their study of mathematics will be lacking in purpose for them.

We recently asked the head of the department of mathematics in a well known institution what he thought could be done to make the study of mathematics more effective and more enjoyable in colleges. He said "Make it interesting." When asked for suggestions for making mathematics interesting he replied in effect: You cannot tell any body how to teach or to make a subject interesting. If the teacher has not sense enough to do it for himself there is not much you can do about it. Now is this so, or is it an opinion based on ignorance of possibilities? At any rate it is a suggestion for those who employ and promote teachers.

A few months ago a physician asked us why mathematics is considered so difficult, and why it is unpopular with many pupils. He said he had seen teachers in the army camps make the subject interesting and almost popular. He expressed an opinion that it is desirable to make it so in schools. We replied, off hand whether right or wrong, that few teachers had tried seriously to make mathematics interesting, or had considered the desirability of selling their goods. In fact we have known teachers to be threatened with involuntary resignation for attempting to do just this thing.

We have observed a number of cases where students came to class with an avowed lack of interest in mathematics, but on coming in contact with a teacher who could make the subject appear to be useful and interesting these students became enthusiastic about the subject. Some of these cases were undergraduates and some were

experienced teachers in public schools taking graduate work. It is not to be inferred that mathematics can be sold to every body. Is it not desirable to leave a "good taste in the mouth" with those who must take mathematics? Have mathematics teachers sounded the depths of the possibilities? Will they ever do so? We wonder.

Schools of education have so far failed to inspire the confidence of teachers in general, and especially college teachers. There are good reasons. As yet there cannot be said to be a real science of education notwithstanding the fact that thousands of degrees in Bachelor of Science in Education have been conferred. We can cite many conferences with students, both undergraduate and graduate, in which they frankly admit that they register for, and get credit for two, three, or more different courses in education and find that preparation for one is sufficient for all, on account of the duplication. This situation is not the exception among institutions. These same students look upon the practice with disapproval, but with a knowing smile and a shrug they say "We must have a certain number of credits in education to be certificated to teach." Our experience in this line corroborates the findings of the committee of the A. A. U. P. who investigated the subject recently, when it was suggested that courses in education be made prerequisite to all college teaching. Whose courses should be taken?

Now let us try to learn something from history (Apologies to Henry Ford) This is seldom done, and much loss to civilization results from this attitude on our part. What is necessary in any science before it is a reasonable business proposition? When did chemistry become a science? When Dalton put it in the way of becoming an exact science, i. e. logical and mathematical. When did electrical knowledge become a science? When its facts and principles were organized into a logical mathematical coherence. This was attained rather less than a century ago. It was not until that was accomplished that we could begin having all the wonderful electrical conveniences in our civilization. Now the proponents of educational theory have made experiment after experiment and have collected data enough (one might think) to make a card index to all who cross the River Styx. But so far no actual science has resulted. Educationists so far have not been able to organize their material, valuable though it may be, into a consistent logical system that can be relied upon to give general public service. One of several things seems inevitable. Either educationists must take up the study of logic and

mathematics to enable them to put over their work as a real science, or else mathematicians must study educational data and organize the material into a science. The third alternative, and the one which we propose is that a friendly constructive co-operation be cultivated, in which neither party is to assume an attitude of omniscience upon the subject.

Mathematicians have furnished many defenders of the traditional claims and purposes of mathematics, but very few have considered the modern psychological phase of its teaching. The educationist camp has furnished many who in the recent past prepared the coffin and shouted and prayed for the final obsequies, but the expected corpse, though much weakened, refused to stop breathing. There have been few educationists who have given serious study to mathematics. Few of either camp have thought of becoming accomplices of those of the other in the same worthy project, namely, the construction of a real science of education. The social sciences in general might be invited to join the party. For they along with education have made little progress toward organizing their material into a public service institution of reliable type. It is noticeable but regrettable that many professors who teach social sciences avoid exactness and mathematical procedure, instead of encouraging it. Why could not departments of education (and social sciences too) offer such a project as thesis subjects, recommending the candidates to study sufficient mathematics and exact science to furnish the method and view point needed? Ultimately something like a real and serviceable science of education will emerge, and then a requirement of educational subjects as a prerequisite for teaching can be justified and we believe made profitable.

The call is to a disarmament conference and a cultivation of friendly relations in the spirit of active constructive co-operation. George will not do it for us. Several years ago we attempted to make a small beginning in this direction. This journal was kind enough to publish our effort after it had been refused by a well known teacher's journal as too theoretical. At that time we found the problem interesting and even tantalizing. We still stand ready to cooperate in a beginning along these lines in any way in which we are able.

We might further suggest that educationists study at least two years of college mathematics and also Young's Fundamental Concepts of Algebra and Geometry and other books on this line before they flatly deny the value of mathematics or prescribe methods for teaching it. The mathematics teachers would do well to read carefully a standard

first book in psychology and something on general method before they condemn outright all educational studies. Both parties would do well to acquaint themselves with the fundamentals of one or more of the so called exact sciences to get the view point of these more exact of the natural sciences.

A History of the Development of Mathematics in the Field of Economics

By Ida Bell Shaw
Ruston, La.

(Continued from Vol. 8, No. 2, November, 1933)

CHAPTER III.

The Mathematico-Economy of the Nineteenth Century

"Of course," says Irving Fisher, "I was not the first to have such dreams for economics as a science. Cournot, Nevons, Walras, Pareto, Marshall, Edgeworth and Wicksell, and several others had dreamed those dreams before me. But some of these had too often found that the market for their wares was small and discouraging."[1]

The mathematical school was inaugurated by M. A. A. Cournot, in 1838, in France. He was the first person with a competent knowledge of both Mathematics and Economics to apply, or to try to apply, mathematics as a treatment of economics questions. His most important works are: *Recherches sur les Principes mathematiques de la Theorie de Richesses* (1838), *Principes de la Theorie des Richesses* (1863), and *Ricerche interno ce principie mathematice della teorica delle richezza* (1875). During his lifetime his books were not noticed. "Even Cournot in those days (days of Jevons and Marshall) was almost unknown among economists and had only barely been rescued from oblivion by Jevons and Marshall. Like the now famous Mendel in Biology, Cournot in economics is an example of a scientist little appreciated until after he had died, and his work had, for a time been forgotten."[2] So we see that the first attention drawn to his books

[1] Irvin Fisher, "Statistics in the Service of Economics," *Journal of the American Statistical Association*, p. 2.
[2] *Ibid.*, p. 2.

was brought about by W. S. Jevons and Albert Marshall. Since that time his principles have been carried through as the basis of our present mathematical development of economics. Our writers today, such as Marshall, Moore, and Fisher, use Cournot's mathematical principles as the basis of their works. What better test is there for the value of one's work ?

"The mathematical approach to economic theory bifurcated at an early point in its course. In 1874, when Leon Walras had just published the early installment of his demonstration of the theory of general equilibrium, he wrote a letter to the aged Cournot which contained a sentence marking the divergence of the two roads of economic theory:

> " 'Notre methode est la meme, ear la mienne est la votre, seulement vous vous placez immediatement an benefice de la loi des grande nombres et sur le chemim qui mene aux applications numeriques. Et moi, je demeure en dica de cette loi sur le terrain des donneis rigeureuses et la pure theorie'."[3]

Both these men employed the mathematical method, but Cournot's method seemed adapted to lead, through the use of the theory of probabilities to numerical applications: Walras tried the solution of a central problem in pure theory, and did not try to get an empirical test of the adequacy of his theoretical construction in the interpretation of the world of economic theory.

The interdependence of the parts of the economic system, and the difficulties with which the economists are confronted, was clearly understood by Cournot, for he stated: "So far we have studied how for each commodity by itself, the law of demand, in connection with the conditions of production of that commodity,[4] determines the price of it and regulates the incomes of its producers. We considered as given and invariable the prices of the other commodities and the incomes of their producers. We considered as given and invariable the prices of other commodities and the incomes of other producers; but in reality the economic system is a whole of which all the parts are connected and reach on each other. An increase in the income of the producers of commodity A will affect the demand for the commodi-

[3]Henry L. Moore, *Synthetic Economies*, Introduction, p. 1. "Our method is the same, for mine is yours, only you make immediate use of the law of large numbers and place yourself on the road leading to numerical applications. And I keep on this side of that law on the ground of rigorous data and pure theory."

[4]*Ibid.*, p. 2.

ties B, C, etc., and the incomes of their producers, and, by its reactions, will involve a change in the demand for commodity A. It seems, therefore, as if, for a complete and rigorous solution of the problems relative to some parts of the economic system, it were indispensable to take the entire system into consideration. But this would surpass the powers of mathematical analysis and of our practical methods of calculation, even if the values of all the constants could be assigned to them numerically."[5]

Cournot began the clearing up of economic theory by his effective use of mathematical methods in his treatment of the law of demand. The quantitative treatment of the conception that lies at the basis of elasticity of demand originated with him. The setting up of his formulae that "demand is a function of the price," marks the inauguration of the calculus.

"According to Cournot, if D equal F(p) is the symbolic expression of the relation between the quantity of commodity demanded, D, and the price per unit of commodity, p; then for many problems in economics it is of importance to know for what value of p the product of pF(p) is a maximum. One of Cournot's concrete illustrations is that of a monopolist, owning a mineral spring where the cost of production is negligible, who wishes to know what price of the commodity will yield him the largest monopoly return. The mathematical condition of a maximum return is

$$\frac{d[p\,F(p)]}{dp} = F(p)+p\,F(p) = 0 \tag{1}$$

The root of equation (1) is the price that will afford the maximum profit. In order to carry this problem to a concrete solution, we must know the empirical form of D equal F(p)."[6] Of this statistical problem Cournot makes the following comment:

> "We may admit that it is impossible to determine the function F(p) empirically for each article, but it is by no means the case that the same obstacles prevent the approximate determination of the value of p which satisfies equation (1) or which renders the product pF(p) a maximum. The construction of a table, where these values could be found, would be the work best cal-

[5]Augustin Cournot, *Researches into the Mathematical Principles of the Theory of Wealth, Bacon's* Translation, p. 127.

[6]Henry L. Moore, *Synthetic Economies*, p. 33.

culated for preparing for the practical and rigorous solution of questions relating to the theory of wealth.

"But even if it were impossible to obtain from statistics the value of p which should render pF(p) a maximum, it would be easy to learn, at least, for all articles to which the attempt has been made to extend commercial statistics, whether current prices are above or below this value. Suppose that when the price becomes the annual consumption as shown by statistics becomes $D - \Delta D$."[7]

According to $\Delta D / \Delta p <$ or $> Dp$, the increase in price, Δp, will increase or diminish the product of pF(p); and, consequently, it will be known whether the two values p and $p + \Delta p$ (assuming Δp to be a small fraction of p) fall above or below the value which makes the product under consideration a maximum.[8] The method of reaching the inequality just discussed may be indicated:

The increase in price will increase the gross receipts if

$$(p+\Delta p)(D-\Delta D)>pD$$

or

$$pD-p\,.\,\Delta D+D\,.\,\Delta p-\Delta p\,.\,\Delta D>pD$$

or

$$-p\,.\,\Delta p+D\,.\,\Delta p-\Delta p\,.\,\Delta D>=0$$

or

$$\Delta D(p+\Delta p)<D\,.\,\Delta p$$

or

$$\frac{\Delta D}{\Delta p} \angle \frac{D}{p+\Delta p}$$

or when Δp is small as compared with p,

$$\frac{\Delta D}{\Delta p} \angle \frac{D}{p} \qquad {}^{9}$$

[7]Augustin Cournot, *op. cit.*, p. 53.
[8]*Ibid.*, p. 54.
[9]Henry L. Moore, *op. cit.*, p. 35.

Definitions of Probability

By Edward S. Allen
Iowa State College

Of the various definitions on which I wish to report, the most widely accepted, of recent years, is the frequency definition. According to it, the probability that, if O occur, P will also happen, is the limit, as the total number of cases considered becomes infinite, of the ratio of the number of cases in which we have O and P to the number in which we have O. There is, I think, no harm in including the possibility that there are but a finite number of terms in the series. It will be noticed that P can have no probability in itself, and that probability, as here defined, always refers to a particular ordered sequence. There is—to illustrate the first point—no sense in saying, "the probability that a good die fall with the 3 uppermost is 1/6". We must precede that with the statement, "if the die is thrown." Should we change the conditional clause so as to read, "if the die has 3 uppermost when 1 centimeter above the table," there may well be a probability, but it will not be 1/6.

According to the definition just given, a probability is that of an event. Keynes, on the other hand, is emphatic in saying that only propositions have probability. A bridge between these two points of view was offered by Ancillon as early as 1794, when he said, "To state that a fact is probable is to state that a proposition is probable"—the probability that an event will happen is the probability that the prediction of it will be fulfilled.

A more important divergence of Keynes' theory from the frequency theory with which we started is his psychological point of view, which leads to an interesting kind of topological probability. The psychological point of view was not, of course, new with Keynes. Laplace defined probability as "belief in the truth of a statement." To illustrate this definition, let us consider the probabilities that (a) it rain tomorrow, (b) it rain tomorrow and then snow, (c) it rain tomorrow and then clear, (d) that a random Fourier series converge to the function from which it was derived. Keynes would, I presume, refuse to assign a numerical probability to any of these events; nevertheless, he would have to give (a) a higher rank than (b) or (c), while holding himself entitled to refuse to compare any other pair named.

Ramsey, who goes even farther than Keynes in his psychological bias, proposes a method for measuring the degree of belief which he identifies with probability. Let us see how he defines a probability of $\frac{1}{2}$, by a combination of beliefs and preferences. We will suppose that you definitely prefer sauerkraut to cheese. You are told, "Either you are to have cheese if a coin, when tossed, comes up heads and kraut if it comes up tails, or the other way about. Which shall it be?" If you then say, "I don't care", that proves that, in your opinion, the probability of appearance of heads is $\frac{1}{2}$. But, after all, does not this indifference signify a belief that, in the long run, heads and tails will appear equally often?

The classical definition of probability was given, perhaps for the first time, by Pascal. It is the "ratio of the number of favorable cases to the number of equally possible (or probable) cases." A fundamental difficulty with this theory is the need for a theory of probability for the definition of "equally possible." It has been proposed that complete ignorance be the basis for "equal possibility"; it has been said that we should have a considerable body of knowledge, but some ignorance. But, after all, this is a rather vague foundation for such a precise number as .5; it seems to me better to return to the first definition given in this account, and to say that two events are equally possible if the ratio of the numbers of their occurrences approaches 1.

Of late years, the vogue of axiomatization has reached the theory of probability. Among the most important of the systems built up from independent postulates are those of Reichenbach and of Tornier. The former associates with two occurrences O and P a number called the probability from O to P:—a member which, however, is given no definition in terms of the objective world. The axioms are those of uniqueness, normalization, addition and multiplication. The axiom of normalization, for example, assigns to certain implication the number 1, and debars negative probability. That of multiplication,—"the product of the probabilities from O to P and from O and P to Q is the probability from O to P and Q." These suffice as a foundation for the whole calculus.

Tornier's theory depends on the topology of null spaces. For him a point is a sequence of elements of unspecified nature. The space of such points has metric character; for the distance between two points can be satisfactorily defined as the reciprocal of the number of the first place at which the sequences differ. Certain point-sets in

this space—in particular, the sets of points whose sequences agree up to a certain place—can be assigned measures and,—with appropriate normalization—the measures can be called probabilities. In more familiar language, we have the probability that a given sequence of events be followed by certain others.

Noteworthy in both these theories is their caution. Reichenbach does not state that a probability is associated with every pair of occurrences (it would be better to say, types of occurrences); Tornier does not give assurance that every set in his null-space has a measure. Both sets of axioms, on the other hand, are consistent with the concept of probability as the limit of relative frequency,—wherever that limit exists.

At this point, however, comes a doubt as to the exact applicability of any theory to nature. We can never know that the fraction of throws of a die resulting in a 3 approaches the limit 1/6—or any limit whatever. What is, then, the validity of any calculus of probabilities? Mathematically, its deductions can be correct. Physically, it can be justified but partially, just as any statement as to the outer world is edged with uncertainty.

Is it conceivable that a new definition of physical probability could lay better claim to correspondence with reality, and at the same time fit into our mathematical theory? Were we to say that a probability is not a limit of frequency within a sequence of events, but the sequence itself, a mathematical theory could be built up, quite equivalent to the current ones—in return for its greater claim to certain truth, it would have lost all usefulness.

It has been suggested that a probability might be, not a number, but an interval. The mechanical difficulties of a corresponding theory would be much greater. What is worse is the retention of conceptual difficulties. When we say "an interval," we mean, of course, an interval with well-defined ends; and a doubt as to the existence of a frequency-limit leads inevitably to doubt as to the physical significance of any definite numbers—other than 0 and 1.

In sum, all theories which have been constructed, if they have physical meaning at all, can be interpreted as frequency theories. The inadequacy of even this interpretation can not be overcome; but the

practical benefits coming from its use justify taking it as a working hypothesis.

REFERENCES

Ancillon—*Doutes sur les bases du calcul des probabilities*, Mem. Ac. Berlin, 1794-5.

Laplace—*Essai philosophique des probabilites*, Paris, 1814.

Keynes—*A treatise on probability*, London, 1921.

Ramsey—*The foundations of mathematics, and other logical essays*, London and New York, 1931.

Reichenbach—*Axiomatik der Wahrscheinlichkeitsrechnung Mathematische Zeitschrift*, vol. 34, 1932.

Tornier—*Grundlagen der Wahrscheinlichkeitsrechnung Acta Mathematica*, vol. 60, 1933.

Magic Squares

By W. VANN PARKER

This note is concerned with a certain type of magic square which I have found to be quite interesting, and which I hope may prove of interest to readers of the News Letter. Such squares and the facts concerning them are perhaps well known, but I do not recall having seen a discussion of them. I shall give two examples and discuss some of the characteristics of the squares.

A magic square with 25 numbers is

11	24	32	45	53
42	55	13	21	34
23	31	44	52	15
54	12	25	33	41
35	43	51	14	22

and one with 49 numbers is

11	25	32	46	53	67	74
52	66	73	17	24	31	45
23	37	44	51	65	72	16
64	71	15	22	36	43	57
35	42	56	63	77	14	21
76	13	27	34	41	55	62
47	54	61	75	12	26	33

Some of the characteristics of such a square are:

I. It contains all two digit numbers formed from the digits 1,2,3, . . ,n. (n = 5 or 7 in the examples above.)

II. The first digit is not duplicated in any row, column, or diagonal.

III. The second digit is not duplicated in any row, column, or diagonal.

IV. The first and second digits are alike for the numbers in the principal diagonal.

V. The sum of the numbers in each row, column and diagonal is $11n(n+1)/2$.

Since we are to form two digit numbers with the integers 1,2,3,..,n, taken in pairs, n cannot exceed 9, and I shall show presently that the squares only exist for n = 5 and n = 7.

Suppose that instead of integers we consider n symbols arranged in a definite order and try to arrange the n^2 permutations of these in a square so as to satisfy I, II, III, and IV above. (V has no meaning now.) Then when n is less than 10 and the symbols are interpreted as integers, we have the magic squares above. We

shall refer to a symbol as even or odd according as the number indicating its position in the ordered arrangement is even or odd.

The method used for constructing the squares is as follows:

1. Write the symbols occupying first position in order in the first row.
2. Write the first symbol occupying first position for each succeeding row under the $(1+d)$th symbol of the row above it, and write the symbols in order toward the right continuing from the left when the end of a row is reached.
3. Write the symbols occupying second position in order in the first column.
4. Write the first symbol occupying second position for each succeeding column to the right of the $(1+d)$th symbol of the column to the left of it, and write the symbols in order toward the bottom continuing from the top when the end of a column is reached.

Let us examine the above characteristics now and see what restrictions must be placed on n and d in order for I,II,III, and IV to be satisfied. If the square is formed according to the rule above, I and IV will be satisfied if d is not a multiple of n, and III will be true whenever II is true. We may restrict ourselves, therefore, to the symbols occupying first position and find under what conditions II is true.

In the discussion to follow we shall assume numbers greater than n to be reduced modulo n, and negative numbers to be increased modulo n.

A. It is obvious that we must have $1<d<n-1$, and hence $n>3$.

B. *d must be prime to n.* For if d is not prime to n there exist integers s and $t,s<n$ and $t<d$, such that $sd=tn$, and hence the first symbol will be in the first column again in the $(s+1)$th row. For in the kth row the first symbol will be in the $[1+(k-1)d]$th column.

C. *n must be odd.* For if n is even d must be odd by B, and hence the even symbols in each row will be under the odd symbols of the row above. This will make all symbols in the secondary diagonal even, and hence they will not be all different.

D. *If n is prime and $1<d<n-1$, it is always possible to construct the square.* It is sufficient to show in this case that the first symbol does not occur again in the first column nor in the principal

diagonal and that the nth symbol does not occur again in the secondary diagonal.

In the kth row, the symbol in the first column is the $[n-(k-1)d+1]$th, the symbol in the principal diagonal is the $[n-(k-1)d+k]$th, and the symbol in the secondary diagonal is the $[n-(k-1)d-k+1]$th.

(a) For the first symbol to appear again in the first column of the $(s+1)$th row, we must have

$$n-sd+1 \equiv 1 \pmod n \text{ or } sd = \lambda n, \left\{ \begin{matrix} s<n, \ \lambda<s \\ d<n, \ \lambda<d \end{matrix} \right\};$$

but this is impossible if n is prime.

(b) If the first symbol appears again in the principal diagonal in in the $(s+1)$th row, we must have

$$n-sd+s+1 \equiv 1 \pmod n \text{ or } (d-1)s = \mu n, \left\{ \begin{matrix} s<n, \ \mu<s \\ d<n, \ \mu<d-1 \end{matrix} \right\},$$

which is impossible if n is prime.

(c) If the nth symbol appears again in the secondary diagonal in the $(s+1)$th row, we must have

$$n-sd-s \equiv 0 \pmod n \text{ or } s(d+1) = \gamma n, \left\{ \begin{matrix} s<n, \ \gamma<s \\ d<n, \ \gamma<d+1 \end{matrix} \right\}$$

which is impossible if n is prime.

It is not necessary that n be prime in order to construct the square. For example, if $n=25$ we may construct the square by taking $d=2$.

We will now show that 5 and 7 are the only numbers less than 10 for which it is possible to construct the square. All values of n except 5, 7, and 9 are excluded by A and C. If $n=9$ the values of d which are multiples of 3 are excluded by (a), the values of $d-1$ which are multiples of 3 are excluded by (b), and the values of $d+1$ which are multiples of 3 are excluded by (c). There are therefore, no values for d when $n=9$.

Book Review Department

Edited by
P. K. Smith

Tables of Integrals and Other Mathematical Data, by *Herbert Bristol Dwight*. New York, Macmillan, 1934. viii plus 222 pp.

The material in this book is arranged under the following headings: algebraic functions (65 pp.), trigonometric functions (34 pp.), inverse trigonometric functions (11 pp.), exponential functions (4 pp.), error function (1 p.), logarithmic functions (13 pp.), hyperbolic functions (14 pp.), inverse hyperbolic functions (10 pp.), elliptic functions (6 pp.), Bessel functions (11 pp.), surface zonal harmonics (2 pp.), definite integrals (8 pp.), differential equations (4 pp.) tables of numerical values (35 pp.).

The section on algebraic functions contains, among other things, the explicit development of the binomial series $(1+x)^n$ for several values of n with a statement of the interval of convergence, a list of factorials and their logarithms up to 11, Bernouilli's numbers and their logarithms up to B and Euler's numbers and their logarithms up to E_7, as well as formulas of differentiation and integral tables. Nine graphs illustrate the relation between integral and integrand.

The integrals of trigonometric functions are preceded by trigonometric identities, Euler's formula, DeMoivre's formula and applications, formulas for plane triangles, trigonometric series, and the derivatives of the fundamental trigonometric functions. One figure is used to illustrate the relation between csc x and its integral.

Inverse trigonometric functions are introduced by their series developments (with the necessary restrictions clearly stated) and by their derivatives. There are 74 integrals involving inverse trigonometric functions given.

Logarithmic functions start out with several series expansions and with the definition of log z for z complex. The gudermannian of x serves as a transition topic from integration involving logarithms to hyperbolic functions. The material under hyperbolic functions and inverse hyperbolic functions parallels that appearing under trigonometric functions. The hyperbolic functions are defined for the complex variable z.

The pages devoted to elliptic functions with associated identities, series and integrals cover all that is needed for the usual applications in theoretical mechanics.

Under Bessel functions are found series, identities, asymptotic expansions, recurrence formulae and ten integrals.

Surface zonal harmonics $P_m(\mu)$, $m=0, 1, 2, \ldots, 7$ and their derivatives are given explicitly. The general formula, recurrence relations for $P_m(\mu)$ and the asymptotic values of $P_m(\cos \Theta)$, m large, conclude the subject matter in this section.

The tables of definite integrals cover several integrals that can be evaluated conveniently only by special devices such as contour integrals and differentiation with respect to a parameter.

A summary of the methods of solving ordinary differential equations as well as the linear partial differential equation of the first order constitutes the section under differential equations.

The appendix contains the following numerical tables:

$\sqrt{a^2+b^2}/a$
Trigonometric functions and their logarithms (4 place)
Radians to degrees, minutes, seconds
Degrees, minutes, seconds to radians
Gamma function (argument from 1.00 to 2.00, 5 place)
Logarithms to base 10 (4 place)
Natural logarithms (1.00 to 2.00, 2.0 to 10.0, 4 place)
Exponential and hyperbolic functions (1.00 to 3.00, 3.05 to 10.00)
Complete elliptic integrals of first kind
Complete elliptic integrals of second kind
Bessel functions
The appendix ends with a list of 37 references (mostly texts).

This book should be at the disposal of all students taking pure or applied mathematics after a first course in the calculus. The author has taken pains to indicate consistently numerical values, e. g. $\log|x|$. and to specify precisely the determination of multi-valued functions used (something most textbooks on calculus studiously avoid). The arrangement of the tables is convenient and effective. Cross-references are given to other formulas in the tables and indications are made as to where more complete discussions or tables may be found. At a first reading the reviewer finds only one minor objection—the smallness of the type in the numerical tables.—*W. E. Byrne.*

Problem Department

Edited by
T. A. Bickerstaff

This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and to solve problems here proposed.

Problems and solutions will be credited to their authors.

Send all communications about problems to T. A. Bickerstaff, University, Mississippi.

LATE SOLUTIONS

An algebraic solution for number 46 by the proposer, C. D. Smith, Mississippi State College.

Numbers 49, and 50 by Mrs. Alta H. Samuels, L. S. U.

PROBLEMS FOR SOLUTION

No. 55. Proposed by Richard A. Miller, University of Mississippi.

To construct an equilaterial triangle with its vertices lying on three given parallel lines.

No. 56. Proposed by Henry Schroeder, Louisiana Polytechnic Institute.

A circle with radius a moves with its center on the circumference of an equal circle, and keeps parallel to a given plane which is perpendicular to the plane of the given circle. Find the volume of the solid it will generate.

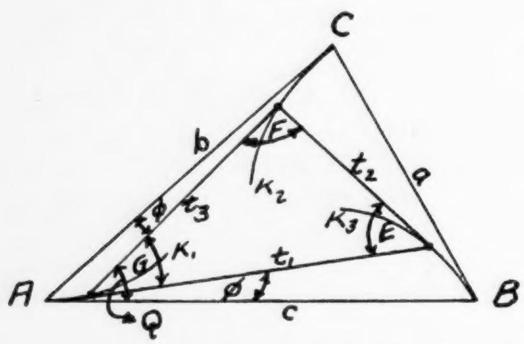
SOLUTIONS

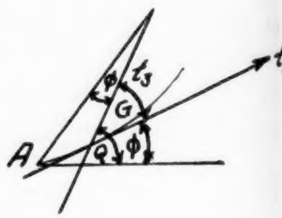
No. 44. Proposed by T. A. Bickerstaff.

A, B, and C are stationed at the three vertices of an oblique triangle. Starting simultaneously, A moves toward B. B moves

toward C, and C moves toward A; all move at the same rate. Investgate their paths.

Solution by A. W. Randall, Prairie View State College, Prairie View Texas.





Let K_1, K_2, and K_3 be the desired paths of A, C, and B respectively. Also let the sides a, b, and c be tangent to the paths K_1, K_2, and K_3 at B, C, and A respectively. Now all subsequent moving of A, B, and C to points A^1, B^1, and C^1 on their respective paths will be tangent to the variable lines t_1, t_2, and t_3 respectively. From the conditions of the problem, we assume that A^1 moves towards B^1, B^1 towards C^1, and C^1 towards A^1, just as they did in the *beginning*, and that at all times the angles which t_1, t_2, and t_3 make with their respective sides (extended if necessary) of the given triangle shall be equal to Φ. From this assumption we shall prove $A=G$; $B=E$ and $C=F$. If t_1, t_2, and t_3 made different angles with the sides of the given triangle we could not prove $A=G$, $B=E$, and $C=F$, and our original state of affairs would be lost. We now set out to prove our assumption.

From the figure,

$$Q-\Phi=G, \qquad (1)$$

$$Q-A=\Phi \qquad (2)$$

Eliminating, $Q-\Phi$ from (1) and (2), we have

$$A=G,$$

and similarly,

$$B=E; \text{ and } C=F,$$

which shows that the variable triangle $A^1B^1C^1$ is similar to triangle ABC. Hence for all positions of triangle $A^1B^1C^1$, angles G, E, and F are constant and respectively equal to angles A, B, and C.

Here it is easily seen that each side of triangle $A^1B^1C^1$ performs the double duty of being tangent to one path, and the radius vector to another in the same immediate counter clockwise direction. Hence each vertex of triangle $A^1B^1C^1$ becomes an INSTANTANEOUS pole for one of the paths.

Therefore each path K_1, K_2, or K_3 is given respectively the following differential equations:

$$dt_3/t_3 = d\Phi \,.\, \cot A$$

$$dt_2/t_2 = d\Phi \,.\, \cot C$$

$$dt_1/t_1 = d\Phi \,.\, \cot B.$$

Integrating we have,

$$\text{Log } t_3 = \Phi \,.\, \cot A + \log H_1, \quad \dots\dots\dots \quad (3)$$

$$\text{Log } t_2 = \Phi \,.\, \cot C + \log H_2, \quad \dots\dots\dots \quad (4)$$

$$\text{Log } t_1 = \Phi \,.\, \cot B + \log H_3. \quad \dots\dots\dots \quad (5)$$

Adding (3), (4), and (5) and reducing, we obtain

$$t_1t_2t_3 = H_1H_2H_3\, e^{\Phi(\cot A + \cot B + \cot C)}, \quad \dots\dots\dots\dots \quad (6)$$

where H_1, H_2, and H_3 are constants of integration, and e is the base of the Naperian system of logarithms.

From the initial conditions of our problem, we determine the values of H_1, H_2, and H_3 as b, a, and c respectively. Hence we may write (6) as

$$t_1t_2t_3 = abce^{\Phi(\cot A + \cot B + \cot C)}, \quad \dots\dots\dots\dots\dots \quad (7)$$

which is the equation for the simultaneous motion of the particles A, B, and C with respect to instantaneous poles and polar axes. But for more practical operations, we desire paths generated from fixed

poles, and we proceed as follows to obtain three different equations for the paths in question.

From similar triangles

$$t_1 = c/b \,.\, t_3, \qquad \text{(8)}$$

$$t_2 = a/b \,.\, t_3. \qquad \text{(9)}$$

Multiplying (8) and (9), we get

$$t_1 t_2 = ac/b^2 \,.\, t^2_3 \qquad \text{(10)}$$

Substituting (10) in (7) and reducing, we obtain

$$t^3_3 = b^3 \,.\, e^{\Phi(\cot A + \cot A + \cot C)}$$

$$t_3 = b \,.\, e^{\Phi\, 3(\cot A + \cot B + \cot C)}, \qquad \text{(11)}$$

which is the equation of the path of K_1, and similarly

$$t_2 = a \,.\, e^{\Phi/3(\cot A + \cot B + \cot C)}, \qquad \text{(12)}$$

and

$$t_1 = c \,.\, e^{\Phi/3(\cot A + \cot B + \cot C)}, \qquad \text{(13)}$$

are the equations of the paths of K_2, and K_3 respectively.

In (11), (12), and (13), we choose the vertices C, B, and A as poles, with sides b, a, and c as polar axes in the order named.

It will also be observed that from and after a certain small triangle $A^1B^1C^1$ within triangle ABC, the points A^1, B^1, and C^1 begin to separate in the same manner as they approached one another—this giving triangle $A^1B^1C^1$ an infinite expansion.